

On the displacement function of isometries of Euclidean buildings

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Abstract

In this note we study the displacement function $d_g(x) := d(gx, x)$ of an isometry g of a Euclidean building. We give a lower bound for $d_g(x)$ depending on the distance from x to the minimal set of g . This answers a question of Rousseau [Rou01, 4.8] and Rapoport-Zink [RZ99, 2.2].

Keywords. Euclidean buildings; displacement function; semisimple isometries; CAT(0) spaces

1 Introduction

Let (X, d) be a metric space and let $g \in \text{Isom}(X)$ be an isometry of X . The *displacement function* $d_g : X \rightarrow [0, \infty)$ of g is the function given by $d_g(x) := d(gx, x)$. The infimum $\delta_g := \inf\{d_g(x) \mid x \in X\}$ of this function is called the *displacement* or *translation length* of g . Let us denote with $\text{Min}_g := \{x \in X \mid d_g(x) = \delta_g\}$ the subset of X where this infimum is attained. We call Min_g the *minimal set* of g . We can divide the isometries of X in three classes depending on the behavior of their displacement functions: we say that g is *elliptic* if it fixes a point, i.e. $\text{Min}_g \neq \emptyset$ and $\delta_g = 0$; *hyperbolic* or *axial* if $\text{Min}_g \neq \emptyset$ and $\delta_g > 0$; and *parabolic* if $\text{Min}_g = \emptyset$. We say that g is *semisimple* if it is elliptic or hyperbolic.

Suppose now that (X, d) is a complete CAT(0) space. Let $C \subset X$ be a closed convex subset. Then the *nearest point projection* $p = p_C : X \rightarrow C$ is a well defined 1-Lipschitz map. It holds that the angle $\angle_{p(x)}(x, z)$ at $p(x)$ between x and z is $\geq \frac{\pi}{2}$ for all $z \in C$ and if $y \in X$ lies on the segment $xp(x)$ between $x \in X$ and its projection $p(x)$, then $p(y) = p(x)$ (see e.g. [BH99, Prop. II.2.4]). Let $g \in \text{Isom}(X)$ be a semisimple isometry, then the convexity of the distance function implies that the minimal set Min_g is a closed convex subset and the displacement function is an increasing function of the distance to the minimal set. More precisely, let $p : X \rightarrow \text{Min}_g$ be the nearest point projection. Suppose $\rho : [0, \infty) \rightarrow X$ is a geodesic ray with $\rho(0) \in \text{Min}_g$ and $p(\rho(t)) = \rho(0)$ for all t , that is, ρ is orthogonal to Min_g . It follows from the convexity of the distance function and the 1-Lipschitz property of the projection that $d_g \circ \rho$ is a strictly monotonically increasing function. We are interested in understanding the growth of this function in the case when X is a Euclidean building. An upper bound for $d_g \circ \rho$ given by $\delta_g + 2t$ follows easily in general from the triangle inequality. It is also not difficult to give examples where this bound is attained. We are therefore interested in giving lower bounds for the function $d_g \circ \rho$.

Euclidean buildings are a special kind of CAT(0) spaces. They are built up from top dimensional building blocks called *apartments* isometric to a Euclidean space. These apartments are glued together following a pattern given by a Euclidean Coxeter complex (a class of groups of isometries of the Euclidean space generated by reflections). This gluing pattern and a certain angle rigidity in the space of directions at points forces the geometry of Euclidean buildings to have some discreteness nature. It follows that in many cases a geometric property at a point or a segment does not depend on the point or segment in question or even on the Euclidean building itself but only on the type of its Coxeter complex and the relative position of the point or segment with respect to this Coxeter complex.

If $X \cong \mathbb{R}^k$ is a Euclidean space, then the function $d_g \circ \rho$ above is given by $d_g(\rho(t)) = \sqrt{\delta_g^2 + Ct^2}$ where $C > 0$ is a constant depending only on the linear part of g . It is reasonable to expect a similar behavior of this function in the case of Euclidean buildings. We show the following (this result was first conjectured by Rousseau, see [Rou01, 4.8] and [RZ99, 2.2]).

Theorem 1.1. *Let X be a Euclidean building without factors isometric to Euclidean spaces. Let $g \in \text{Isom}(X)$ be an isometry of X . Then*

$$d_g(x) \geq \sqrt{\delta_g^2 + C \cdot d(x, \text{Min}_g)^2}$$

for a constant $C > 0$ depending only on the spherical Coxeter complex associated to X and, if g is hyperbolic, on the type of the endpoint $c(\infty)$ of an axis c of g .

Notice that the conclusion of Theorem 1.1 can only be satisfied by semisimple isometries (see Proposition 2.1), that is, Euclidean buildings do not admit parabolic isometries.

2 Preliminaries

In this paper we consider Euclidean buildings from the CAT(0) viewpoint as presented in [KL97, Section 4], we refer to it for the basic definitions and facts about Coxeter complexes, Euclidean and spherical buildings. For more information on CAT(0) spaces in general we refer to [BH99].

All geodesic segments, lines and rays will be assumed to be parametrized by arc-length.

Let (X, d) be a Euclidean building. Its Tits boundary $\partial_T X$ with the *Tits distance* \angle_T is a CAT(1) space admitting a unique structure (possibly trivial if X is a Euclidean space) as a *thick* spherical building modelled on a spherical Coxeter complex (S, W) . We say that (S, W) is the spherical Coxeter complex associated to X . The *space of directions* or *link* $\Sigma_x X$ at a point $x \in X$ with the *angle metric* \angle is also a CAT(1) space admitting a natural structure as a spherical building modelled on the same spherical Coxeter complex (S, W) , although this structure is not in general thick. The spherical Coxeter complex (S, W) splits off a spherical factor in its decomposition as spherical join (see [KL97, Section 3.3]) if and only if the Coxeter group W has fixed points on the sphere S . Equivalently, if and only if the *model Weyl chamber* $\Delta_{\text{mod}} := S/W$ has diameter $> \frac{\pi}{2}$ (in this case, it actually has diameter π). This spherical factor of the Coxeter complex corresponds to a spherical factor of $\partial_T X$, which in turn corresponds to a Euclidean factor of X in its decomposition as a product of Euclidean buildings. An isometry g of a Euclidean building $X = X' \times \mathbb{R}^k$ where X' is a Euclidean building without Euclidean factors decomposes naturally as (g_1, g_2) where g_1 is an isometry of X' and g_2 is an isometry of \mathbb{R}^k . Since we can completely describe the

displacement function of Euclidean isometries (cf. the paragraph previous to Theorem 1.1 in the Introduction), we restrict our attention to Euclidean buildings without Euclidean factors, or equivalently, with model Weyl chamber Δ_{mod} of diameter $\leq \frac{\pi}{2}$.

For a spherical building B modelled on the spherical Coxeter complex (S, W) there is a natural 1-Lipschitz map $\theta : B \rightarrow \Delta_{\text{mod}}$, which is an isometry on chambers. The image of a point in B under this map is called its *type*. Let Γ denote the isometry group of the model Weyl chamber Δ_{mod} . If (S, W) has no spherical factors, then Δ_{mod} is a spherical simplex and Γ is finite. In this case, the quotient $\Delta_{\text{mod}}/\Gamma$ is again a spherical polyhedron. Let us denote the image of a point in B under the composition $\tau : B \rightarrow \Delta_{\text{mod}} \rightarrow \Delta_{\text{mod}}/\Gamma$ by its *subtype*. Notice that antipodal points have the same subtype.

An isometry of a Euclidean building induces an isometry of its model Weyl chamber. Hence, isometries preserve the subtypes of directions and points at infinity.

If g is a hyperbolic isometry of a CAT(0) space X , then Min_g is isometric to a product $Y \times \mathbb{R}$, where Y is again a CAT(0) space. The isometry g acts on Y as the identity and on \mathbb{R} as a translation of length δ_g . The sets $\{y\} \times \mathbb{R} \subset \text{Min}_g$ are geodesics lines preserved by g and are called *axes* of g . The axes are parallel to each other, thus Min_g is a subset of the parallel set of an axis.

The displacement function of an isometry g of a CAT(0) space (X, d) is a Lipschitz convex function. This kind of functions on CAT(0) spaces are asymptotically linear along any ray. That is, if $\xi \in \partial_T X$ and ρ is a ray asymptotic to ξ , then $\text{slope}_g(\xi) := \lim_{t \rightarrow \infty} \frac{1}{t} d_g(\rho(t))$ exists and does not depend on the particular ray ρ . Let now ρ be a geodesic ray with $\rho(0) \in \text{Min}_g$ and orthogonal to Min_g . The observation above implies that there is a constant $K (= \text{slope}_g(\rho(\infty)))$ depending on g and $\rho(\infty)$, such that $d_g(\rho(t)) \geq \frac{K}{2}t = \frac{K}{2}d(\rho(t), \text{Min}_g)$ for t big enough. Proposition 2.1 makes a more precise statement in the case of Euclidean buildings and shows that we can choose the constant K to depend only on the associated spherical Coxeter complex.

Isometries of Euclidean buildings are always semisimple. This follows from the following result of Parreau [Par00, Theorem 4.1]. We include here a variant of her proof for the convenience of the reader.

Proposition 2.1. *Let X be a Euclidean building without Euclidean factors and let g be an isometry. Let $D_a = D_a(g) = \{x \in X \mid d_g(x) \leq a\}$ denote the sublevel sets of the displacement function. There is a constant $K > 0$ depending only on the spherical Coxeter complex (S, W) associated to X such that if $D_a \neq \emptyset$ then*

$$d_g(x) + a \geq Kd(x, D_a).$$

In particular, all isometries of X are semisimple.

Proof. Suppose there is no such constant K . Then there exist sequences X_n of Euclidean buildings with associated spherical Coxeter complex (S, W) , isometries $g_n \in \text{Isom}(X_n)$, points $x_n \in X_n$ and real numbers a_n such that $\emptyset \neq D_n := D_{a_n}(g_n) \subset X_n$ and

$$d_{g_n}(x_n) + a_n \leq \frac{1}{n}d(x_n, D_n).$$

Let y_n be the projection of x_n onto D_n and extend the segment $y_n x_n$ to a ray $y_n \zeta_n$ with $\zeta_n \in \partial_T X_n$ (see Figure 1). After taking a subsequence, we may assume that the subtype of ζ_n

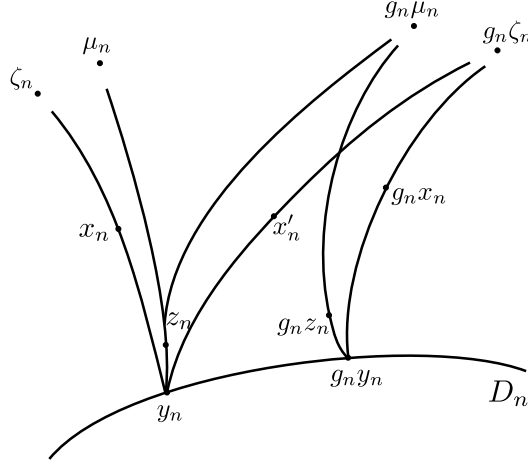


Figure 1: Proof of Proposition 2.1

converges to some $\tau \in \Delta_{\text{mod}}/\Gamma$. Since the distance between two points of the same subtype can take only finitely many values, for n big enough there is a unique point $\mu_n \in \partial_T X_n$ nearest to ζ_n of subtype τ and $\angle_T(\mu_n, \zeta_n) \rightarrow 0$. This implies that $g_n \mu_n$ is also the unique point nearest to $g_n \zeta_n$ of subtype τ .

Let x'_n be the point on the ray $y_n(g_n \zeta_n)$ at distance $d(x_n, y_n)$ from y_n . Then $d(g_n x_n, x'_n) \leq d(g_n y_n, y_n) \leq a_n$ because the rays $y_n(g_n \zeta_n)$ and $(g_n y_n)(g_n \zeta_n)$ are asymptotic (see Figure 1). Hence

$$\frac{1}{n}d(x_n, y_n) \geq d(x_n, g_n x_n) + a_n \geq d(x_n, g_n x_n) + d(g_n x_n, x'_n) \geq d(x_n, x'_n).$$

Considering the triangle (y_n, x_n, x'_n) this implies that $\angle_{y_n}(\zeta_n, g_n \zeta_n) = \angle_{y_n}(x_n, x'_n) \rightarrow 0$. Hence, from the triangle inequality in the link $\Sigma_{y_n} X_n$ we obtain

$$\angle_{y_n}(\mu_n, g_n \mu_n) \leq \angle_{y_n}(\mu_n, \zeta_n) + \angle_{y_n}(\zeta_n, g_n \zeta_n) + \angle_{y_n}(g_n \zeta_n, g_n \mu_n) \rightarrow 0.$$

It follows that for n big enough $\angle_{y_n}(\mu_n, g_n \mu_n) = 0$ because μ_n and $g_n \mu_n$ have the same subtype. Therefore the rays $y_n \mu_n$ and $y_n(g_n \mu_n)$ must initially coincide. This in turn implies that for z_n on the ray $y_n \mu_n$ close enough to y_n (such that z_n also lies on the ray $y_n(g_n \mu_n)$) holds $d(z_n, g_n z_n) \leq d(y_n, g_n y_n) \leq a_n$ since the rays $y_n(g_n \mu_n)$ and $(g_n y_n)(g_n \mu_n) = g_n(y_n \mu_n)$ are asymptotic (see Figure 1). Thus $z_n \in D_n = \{x \in X_n \mid d_{g_n}(x) \leq a_n\}$ and since y_n is the projection of x_n onto D_n we get $\angle_{y_n}(x_n, z_n) \geq \frac{\pi}{2}$. We obtain a contradiction to $\angle_{y_n}(x_n, z_n) = \angle_{y_n}(\zeta_n, \mu_n) \rightarrow 0$. This proves the first part.

The inequality for d_g shows that for a given point $x \in X$ and $d_g(x) \geq a > \delta_g$ the distance of x to D_a is bounded independently of a . In other words, there is a $r > 0$ such that $B_x(r) \cap D_a \neq \emptyset$ for all $a > \delta_g$. This implies that $B_x(r) \cap \text{Min}_g = B_x(r) \cap (\bigcap_a D_a)$ is nonempty and therefore g is semisimple. \square

Remark 2.2. The lower bound for d_g given by Proposition 2.1 is interesting only for big values of $d(x, \text{Min}_g)$: if $\delta_g > 0$ and $d(x, \text{Min}_g) \leq \delta_g/C$ we just get the trivial bound $d_g(x) \geq 0$.

We will also need the following generalization of [KL97, Lemma 4.6.3].

Proposition 2.3. *Let c be a geodesic line in the Euclidean building X with $c(\infty) =: \xi \in \partial_T X$ and denote with P_c the parallel set of c . Let $\alpha > 0$ be the distance of ξ to the boundary of*

the union of all chambers in $\partial_T X$ containing ξ (notice that α depends only on the (sub)type of ξ in Δ_{mod}). Let ρ be a geodesic ray asymptotic to ξ . Then $\rho(t) \in P_c$ for $t \geq \frac{d(\rho(0), P_c)}{\sin \alpha}$.

Proof. Let $A \subset X$ be an apartment containing the ray ρ . The convex hull in $\partial_T X$ of $c(-\infty) \in \partial_T X$ and all chambers in $\partial_T A$ containing ξ is an apartment contained in $\partial_T P_c$, which is the boundary at infinity of an apartment $A' \subset P_c$. It follows that $\partial_T A \cap \partial_T A'$ is a union of chambers containing ξ in its interior. [KL97, Lemma 4.6.3] applied to the regular geodesic lines in A asymptotic to points in $\partial_T A \cap \partial_T A'$ implies that $A \cap A'$ is a convex subset with boundary $\partial_T(A \cap A') = \partial_T A \cap \partial_T A'$ (cf. [KL97, Lemma 4.6.5]). In particular, $A \cap A'$ contains a subray of ρ . That is, $\rho(t) \in P_c$ for t big enough.

Let t_0 be such that $\rho(t_0)$ is the first point of ρ in P_c . We want to show that $t_0 \leq \frac{d(\rho(0), P_c)}{\sin \alpha}$. Let $x = \rho(0)$, $y = \rho(t_0)$ and let z be the projection of x onto P_c . Then $\angle_z(x, y) \geq \frac{\pi}{2}$. Since y is the point where ρ enters P_c we see that $\angle(\vec{y}\vec{x}, \Sigma_y P_c) > 0$. In particular, the chambers in $\Sigma_y X$ containing $\vec{y}\vec{x}$ are not contained in $\Sigma_y P_c$. It follows that $\angle(\vec{y}\vec{x}, \Sigma_y P_c) \geq \alpha$ because $\vec{y}\vec{x}$ is antipodal to $\vec{y}\vec{\xi}$ and therefore both have the same subtype as ξ . This in turn implies that $\angle_y(x, z) \geq \alpha$. By considering a comparison triangle for (x, y, z) we obtain $d(x, y) \sin \alpha \leq d(x, z)$, that is, $t_0 \sin \alpha \leq d(\rho(0), P_c)$. \square

Remark 2.4. The non-existence of parabolic isometries of Euclidean buildings can also be deduced from Proposition 2.3 as follows. By [CL10, Corollary 1.5] an isometry g has a fixed point in $X \cup \partial_T X$. If g is not elliptic, it must fix a point $\xi \in \partial_T X$ at infinity. Let c be a geodesic line asymptotic to ξ . The parallel set of c splits as a product $P_c \cong Y \times \mathbb{R}$ where $Y \subset X$ is again a Euclidean building with $\text{rank}(Y) = \text{rank}(X) - 1$. The isometry g induces an isometry \bar{g} of Y as follows: for $y \in Y$ consider the ray $g(\bar{y}\xi)$ where $\bar{y} := (y, 0)$. Since ξ is fixed by g , this ray is again asymptotic to ξ . Therefore by Proposition 2.3 it eventually coincides with a line parallel to c of the form $\{y'\} \times \mathbb{R}$. The map $y \mapsto y' =: \bar{g}(y)$ is an isometry of Y . By induction on the rank of the building \bar{g} must be a semisimple isometry. If \bar{g} is elliptic and $y \in Y$ is a fixed point, then the rays $\bar{y}\xi$ and $g(\bar{y}\xi)$ must eventually coincide. It follows that there is a geodesic line which eventually coincides with $\bar{y}\xi$ that is translated by g , thus this line is an axis of g and we conclude that g is semisimple. If \bar{g} is hyperbolic and $l \subset \text{Min}_{\bar{g}} \subset Y$ is an axis of \bar{g} , then the rays $(g\bar{y})\xi$ for $y \in l$ eventually enter the two dimensional flat $F = l \times \mathbb{R}$ because l is preserved by \bar{g} . It follows that for $x = (y, t) \in F$ with t big enough $x, gx, g^2x \in F \cap gF$ and these three points lie on a segment because $y, \bar{g}y, \bar{g}^2y \in l$. This implies that there is a geodesic line (not asymptotic to ξ) through x, gx, g^2x that is translated by g and again g is semisimple.

3 Proof of the main result

If g is elliptic, then the result follows from Proposition 2.1.

Let g be a hyperbolic isometry with axis asymptotic to $\xi \in \partial_T X$. That is, if c is an axis of g , then $g(c(t)) = c(t + \delta_g)$ and $c(\infty) = \xi$. Let $P \cong Y \times \mathbb{R}$ be the parallel set of an axis of g . The metric space Y is again an Euclidean building and we denote again with d its induced distance. Its Tits boundary $\partial_T Y$ can be canonically identified with $\Sigma_\xi \partial_T X$. It follows that the spherical Coxeter complex associated to Y depends only on (S, W) and the type of the face $\sigma \subset \partial_T X$ containing ξ in its interior. Y splits off a Euclidean factor of dimension $\dim(\sigma)$. The isometry g restricts to an isometry of P and $\text{Min}_g \cong Y' \times \mathbb{R}$ with

$Y' \subset Y$. Thus, g induces an elliptic isometry \bar{g} of Y with fixed point set Y' . The projection of \bar{g} to the Euclidean factor of Y induces an isometry of the face σ . Therefore there are only finitely many possibilities for the linear part of the projection of \bar{g} to the Euclidean factor and they depend only on the type of σ . Since there are also only finitely many types of faces of $\partial_T X$, Proposition 2.1 gives us a constant $C' > 0$ depending only on (S, W) such that $d(y, \bar{g}y) \geq C'd(y, Y')$ for all $y \in Y$.

Suppose first that $x \in P \cong Y \times \mathbb{R}$ (cf. [Rou01, Prop. 4.4]). Then $x = (y, t) \in Y \times \mathbb{R}$ and $gx = (\bar{g}y, t + \delta_g)$. It follows that

$$d(x, gx)^2 = d(y, \bar{g}y)^2 + \delta_g^2 \geq C'^2 d(y, Y')^2 + \delta_g^2 = C'^2 d(x, \text{Min}_g)^2 + \delta_g^2.$$

This proves the assertion in this case. We want now to generalize this idea.

Let $b_\xi(x) = \lim_{t \rightarrow \infty} (d(x, c(t)) - t)$ be a Busemann function centered at ξ , where c is an axis of g with $c(\infty) = \xi$. For $x \in X$ holds $b_\xi(gx) = \lim_{t \rightarrow \infty} (d(gx, c(t)) - t) = \lim_{t \rightarrow \infty} (d(gx, c(t + \delta_g)) - (t + \delta_g)) = \lim_{t \rightarrow \infty} (d(gx, gc(t)) - t) - \delta_g = \lim_{t \rightarrow \infty} (d(x, c(t)) - t) - \delta_g = b_\xi(x) - \delta_g$. It follows that g maps the horosphere $HS(r) := b_\xi^{-1}(r)$ (the horoball $Hb(r) := b_\xi^{-1}((-\infty, r])$) to the horosphere $HS(r - \delta_g)$ (the horoball $Hb(r - \delta_g)$).

Consider a point $x \in X$. Let x' be the point on the ray $x\xi$ at distance δ_g from x . Then x' and gx are in the same horosphere $HS(b_\xi(gx))$ and x' is the projection of x onto the horoball $Hb(b_\xi(gx))$. It follows that $\angle_{x'}(x, gx) \geq \frac{\pi}{2}$ and from triangle comparison with respect to the triangle (x, x', gx) we conclude

$$d(x, gx)^2 \geq d(x, x')^2 + d(x', gx)^2 = \delta_g^2 + d(x', gx)^2.$$

Hence, it suffices to show that there is a constant $C > 0$ depending only on the spherical Coxeter complex associated to X and the type of ξ such that $d(x', gx) \geq Cd(x, \text{Min}_g)$. In the case $x = (y, t) \in P \cong Y \times \mathbb{R}$ above, we have $x' = (y, t + \delta_g)$. Hence, $d(x', gx) = d(y, \bar{g}y) \geq C'd(y, Y') = C'd(x, \text{Min}_g)$.

Suppose there is no such constant $C > 0$. Then there exist sequences X_n of Euclidean buildings with associated spherical Coxeter complex (S, W) , hyperbolic isometries $g_n \in \text{Isom}(X_n)$ with axes asymptotic to a point $\xi_n \in \partial_T X_n$ of constant subtype $\tau \in \Delta_{\text{mod}}/\Gamma$, and points $x_n \in X_n$ such that

$$d(x'_n, g_n x_n) \leq \frac{1}{n} d(x_n, M_n),$$

where $M_n := \text{Min}_{g_n}$ and x'_n is the projection of x_n onto the horoball $Hb(b_{\xi_n}(g_n x_n))$ centered at ξ_n .

After rescaling the buildings X_n we may assume that $\delta_{g_n} = \delta$ is independent of n . From Proposition 2.1 we get a constant $K > 0$ depending only on (S, W) such that

$$Kd(x_n, M_n) - \delta \leq d(x_n, g_n x_n) \leq d(x_n, x'_n) + d(x'_n, g_n x_n) \leq \delta + \frac{1}{n} d(x_n, M_n).$$

This implies that $d(x_n, M_n) \leq \frac{2\delta}{K-1/n} \leq \frac{3\delta}{K}$ for n big enough.

Let $P_n \subset X_n$ the parallel set of an axis of g_n . Proposition 2.3 implies that there is an α independent of n such that a point on the ray $x_n \xi_n$ at distance $\geq \frac{d(x_n, P_n)}{\sin \alpha}$ from x_n lies in P_n . For $k \in \mathbb{N}$, let x_n^k be the point on the ray $x_n \xi_n$ at distance $k\delta$ from x_n (see Figure 2). Choose a fixed $m \in \mathbb{N}$ such that $m \geq \frac{3}{K \sin \alpha}$. Then x_n^m lies in P_n for all n big enough because $m\delta \geq \frac{3\delta}{K \sin \alpha} \geq \frac{d(x_n, M_n)}{\sin \alpha} \geq \frac{d(x_n, P_n)}{\sin \alpha}$.

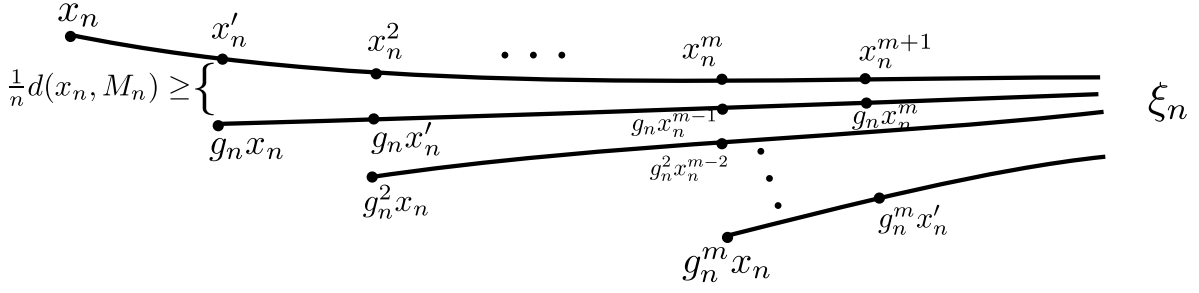


Figure 2: Proof of the main result

We show now inductively on k that $d(g_n^k x_n, x_n^k) \leq \frac{k}{n} d(x_n, M_n)$. For $k = 0$ this is trivial. Let us prove it for $k + 1$. First observe that the rays $x_n^k \xi_n$ and $(g_n^k x_n) \xi_n = g_n^k(x_n \xi_n)$ are asymptotic and therefore $d(x_n^{k+1}, g_n^k x'_n) \leq d(x_n^k, g_n^k x_n)$ (see Figure 2). Hence,

$$\begin{aligned} d(g_n^{k+1} x_n, x_n^{k+1}) &\leq d(g_n^{k+1} x_n, g_n^k x'_n) + d(g_n^k x'_n, x_n^{k+1}) \\ &\leq d(g_n x_n, x'_n) + d(x_n^k, g_n^k x_n) \\ &\leq \frac{1}{n} d(x_n, M_n) + \frac{k}{n} d(x_n, M_n) = \frac{k+1}{n} d(x_n, M_n). \end{aligned}$$

In particular, we have that $d(x_n^m, g_n^m x_n) \leq \frac{m}{n} d(x_n, M_n)$ and this implies

$$d(x_n^m, M_n) \geq d(g_n^m x_n, M_n) - d(x_n^m, g_n^m x_n) \geq d(x_n, M_n) - \frac{m}{n} d(x_n, M_n) = \left(1 - \frac{m}{n}\right) d(x_n, M_n).$$

Since x_n^m lies in the parallel set P_n of an axis of g_n , we already know in this case that there is a constant $C' > 0$ such that $d(x_n^{m+1}, g_n x_n^m) \geq C' d(x_n^m, M_n) \geq C' \left(1 - \frac{m}{n}\right) d(x_n, M_n)$.

On the other hand, since the rays $x'_n \xi_n$ and $(g_n x_n) \xi_n = g_n(x_n \xi_n)$ are asymptotic, we see that $d(x'_n, g_n x_n) \geq d(x_n^{m+1}, g_n x_n^m)$. Thus, we obtain

$$\frac{1}{n} d(x_n, M_n) \geq d(x'_n, g_n x_n) \geq d(x_n^{m+1}, g_n x_n^m) \geq C' \left(1 - \frac{m}{n}\right) d(x_n, M_n).$$

This implies $1 \geq C'(n - m)$, which is a contradiction for n big enough. \square

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